Non-parametric Test

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PATIRA DATA SCIENCE
DISCOVERING KNOWLEDGE FROM DATA

The Ohio State University

Overview

- Distinguish Parametric and Nonparametric Test Procedures
- Explain commonly used Nonparametric Test Procedures
- Perform Hypothesis Tests Using Nonparametric Procedures

Hypothesis Testing

- **Parametric**
  - **TTest**
  - **ANOVA**
- **Non-Parametric**
  - **U-Test**
  - **Kruskal-Wallis**

Overview of Hypothesis testing

Parametric Test Procedures

- Involve population parameters (Mean).
- Have stringent assumptions (Normality).
- Examples: TTest and ANOVA.

Parametric Assumptions

- The observations must be independent
- The observations must be drawn from normally distributed populations

Nonparametric Test Procedures

- Data not normally distributed
- Data measured on any scale (ratio or interval, ordinal or nominal).
- Example: Mann-Whitney U test, Kruskal-Wallis etc.

Mann-Whitney U Test

- Nonparametric alternative to two-sample TTest.
- Actual measurements not used – ranks of the measurements are used.
- Data can be ranked from highest to lowest or lowest to highest values.
- Mann-Whitney U statistic equation. Calculate U and U'.

$$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$

$$U' = n_1 n_2 - U$$

Mann-Whitney *U* Test: Sample Size Consideration

- Size of sample 1: n_1
- Size of sample 2: n_2
- If both n_1 and n_2 are ≤ 20, the small sample procedure is appropriate.
- If either n_1 or n_2 is greater than 20, the large sample procedure is appropriate.

Example of Mann-Whitney U test

- Two tailed null hypothesis that there is no difference between the heights of male and female students
- H_o: Male and female students are the same height
- H_a: Male and female students are not the same height

Example of Mann-Whitney U test

Heights of males (cm)	Heights of females (cm)
170	168
188	173
185	175
183	163
178	165
180	
193	
$n_1 = 7$	$n_2 = 5$

Rank the heights of males and females

Heights of males (cm)	Heights of females (cm)
170	168
188	173
185	175
183	163
178	165
180	
193	
$n_1 = 7$	$n_2 = 5$

Heights of males (cm)	Heights of females (cm)	Ranks of male heights	Ranks of female heights
193	175	1	7
188	173	2	8
185	168	3	10
183	165	4	11
180	163	5	12
178		6	
170		9	
$n_1 = 7$	$n_2 = 5$	$R_1 = 30$	$R_2 = 48$

$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$

$U = (7)(5) + \frac{(7)(8)}{2} - 30$

$U = 35 + 28 - 30$

$U = 33$

$U' = n_1 n_2 - U$

$U' = (7)(5) - 33$

$U' = 2$

The smaller value of U and U' is the one used when consulting significance tables

Heights of males (cm)	Heights of females (cm)	Ranks of male heights	Ranks of female heights
193	175	1	7
188	173	2	8
185	168	3	10
183	165	4	11
180	163	5	12
178		6	
170		9	
$n_1 = 7$	$n_2 = 5$	$R_1 = 30$	$R_2 = 48$

Critical Values of the Mann-Whitney U
(Two-Tailed Testing)

n_2	α	n_1																	
		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	.05	--	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
	.01	--	0	0	0	0	0	0	0	0	1	1	1	2	2	2	2	3	3
4	.05	--	0	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	14
	.01	--	--	0	0	0	1	1	2	2	3	3	4	5	5	6	6	7	8
5	.05	0	1	2	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20
	.01	--	--	0	1	1	2	3	4	5	6	7	7	8	9	10	11	12	13
6	.05	1	2	3	5	6	8	10	11	13	14	16	17	19	21	22	24	25	27
	.01	--	0	1	2	3	4	5	6	7	9	10	11	12	13	15	16	17	18
7	.05	1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
	.01	--	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	22	24
8	.05	2	4	6	8	10	13	15	17	19	22	24	26	29	31	34	36	38	41
	.01	--	1	2	4	6	7	9	11	13	15	17	18	20	22	24	26	28	30
9	.05	2	4	7	10	12	15	17	20	23	26	28	31	34	37	39	42	45	48
	.01	0	1	3	5	7	9	11	13	16	18	20	22	24	27	29	31	33	36
10	.05	3	5	8	11	14	17	20	23	26	29	33	36	39	42	45	48	52	55
	.01	0	2	4	6	9	11	13	16	18	21	24	26	29	31	34	37	39	42
11	.05	3	6	9	13	16	19	23	26	30	33	37	40	44	47	51	55	58	62
	.01	0	2	5	7	10	13	16	18	21	24	27	30	33	36	39	42	45	48

$U = n_1n_2 + \frac{n_1(n_1+1)}{2} - R_1$

$U = (7)(5) + \frac{(7)(8)}{2} - 30$

$U = 35 + 28 - 30$

$U = 33$

$U' = n_1n_2 - U$

$U' = (7)(5) - 33$

$U' = 2$

Heights of males (cm)	Heights of females (cm)	Ranks of male heights	Ranks of female heights
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178		6	
170		9	
$n_1 = 7$	$n_2 = 5$	$R_1 = 30$	$R_2 = 48$

To be statistically significant, the obtained U has to be equal to or less than this critical value.

$U_{0.05(7,5)} = U_{0.05(5,7)} = 5$

As 2 < 5, H_0 is rejected

Mann-Whitney *U* Test: Formulas for Large Sample Case

If either n_1 or n_2 is > 20, the sampling distribution of U is approximately normal.

$$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - W_1$$

where: n_1 = number in group 1

n_2 = number in group 2

W_1 = sum or the ranks of values in group 1

$$\mu_U = \frac{n_1 \cdot n_2}{2}$$

$$\sigma_U = \sqrt{\frac{n_1 \cdot n_2 (n_1 + n_2 + 1)}{12}}$$

$$Z = \frac{U - \mu_U}{\sigma_U}$$

Comparing Three or More Populations:

Kruskal-Wallis *H*-Test

- Tests the equality of more than two (p) population probability distributions
- Corresponds to ANOVA.
- Uses χ^2 distribution with $p - 1$ df

Kruskal-Wallis *H*-Test for Comparing *k* Probability Distributions

H_0: The *k* probability distributions are identical

H_a: At least two of the *k* probability distributions differ in location.

Squared total of each group

Test statistic: $$H = \left(\frac{12}{n(n+1)} \sum \frac{R_j^2}{n_j} \right) - 3(n+1)$$

Kruskal-Wallis *H*-Test for Comparing *k* Probability Distributions

where

n_j = Number of measurements in sample *j*

R_j = Rank sum for sample *j*, where the rank of each measurement is computed according to its relative magnitude in the totality of data for the *k* samples

n = Total sample size = $n_1 + n_2 + . . . + n_k$

Kruskal-Wallis *H*-Test for Comparing *k* Probability Distributions

Rejection region:

$H > \chi^2_\alpha$ with (k – 1) degrees of freedom

Ties: Assign tied measurements the average of the ranks they would receive if they were unequal but occurred in successive order. For example, if the third-ranked and fourth-ranked measurements are tied, assign each a rank of (3 + 4)/2 = 3.5. The number should be small relative to the total number of observations.

Conditions Required for the Validity of the Kruskal-Wallis *H*-Test

1. The *k* samples are random and independent.
2. The *k* probability distributions from which the samples are drawn are continuous

Kruskal-Wallis *H*-Test Procedure

1. Assign ranks, R_i, to the n combined observations
   - Smallest value = 1; largest value = n
   - Average ties
2. Sum ranks for each group
3. Compute test statistic

Squared total of each group

$$H = \left(\frac{12}{n(n+1)} \sum \frac{R_j^2}{n_j} \right) - 3(n+1)$$

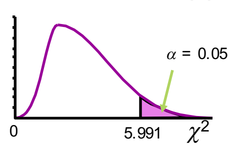
Kruskal-Wallis *H*-Test Example

A production manager wants to see if three filling machines have different filling times. He assigns 15 similarly trained and experienced workers, 5 per machine, to the machines. At the **.05** level of significance, is there a difference in the **distribution** of filling times?

Mach1	Mach2	Mach3
25.40	23.40	20.00
26.31	21.80	22.20
24.10	23.50	19.75
23.74	22.75	20.60
25.10	21.60	20.40

Kruskal-Wallis *H*-Test Solution

- H_0: Identical Distrib.
- H_a: At Least 2 Differ
- α = .05
- *df* = $p - 1 = 3 - 1 = 2$
- **Critical Value(s):**



Kruskal-Wallis *H*-Test Solution

Raw Data			Ranks		
Mach1	Mach2	Mach3	Mach1	Mach2	Mach3
25.40	23.40	20.00			
26.31	21.80	22.20			
24.10	23.50	19.75			
23.74	22.75	20.60			
25.10	21.60	20.40			

Kruskal-Wallis *H*-Test Solution

Raw Data			Ranks		
Mach1	Mach2	Mach3	Mach1	Mach2	Mach3
25.40	23.40	20.00			
26.31	21.80	22.20			
24.10	23.50	19.75			1
23.74	22.75	20.60			
25.10	21.60	20.40			

Kruskal-Wallis *H*-Test Solution

Raw Data			Ranks		
Mach1	Mach2	Mach3	Mach1	Mach2	Mach3
25.40	23.40	20.00			2
26.31	21.80	22.20			
24.10	23.50	19.75			1
23.74	22.75	20.60			
25.10	21.60	20.40			

Kruskal-Wallis *H*-Test Solution

Raw Data			Ranks		
Mach1	Mach2	Mach3	Mach1	Mach2	Mach3
25.40	23.40	20.00			2
26.31	21.80	22.20			
24.10	23.50	19.75			1
23.74	22.75	20.60			
25.10	21.60	20.40			3

Kruskal-Wallis *H*-Test Solution

Raw Data			Ranks		
Mach1	Mach2	Mach3	Mach1	Mach2	Mach3
25.40	23.40	20.00	14	9	2
26.31	21.80	22.20	15	6	7
24.10	23.50	19.75	12	10	1
23.74	22.75	20.60	11	8	4
25.10	21.60	20.40	13	5	3

Kruskal-Wallis *H*-Test Solution

Raw Data				Ranks		
Mach1	Mach2	Mach3		Mach1	Mach2	Mach3
25.40	23.40	20.00		14	9	2
26.31	21.80	22.20		15	6	7
24.10	23.50	19.75		12	10	1
23.74	22.75	20.60		11	8	4
25.10	21.60	20.40		13	5	3
			Total	65	38	17

Kruskal-Wallis *H*-Test Solution

$$H = \left(\frac{12}{n(n+1)}\sum \frac{R_j^2}{n_j}\right) - 3(n+1)$$

$$= \left(\frac{12}{(15)(16)}\left(\frac{(65)^2}{5} + \frac{(38)^2}{5} + \frac{(17)^2}{5}\right)\right) - 3(16)$$

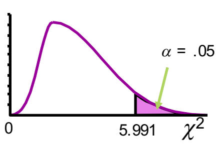
$$= \left(\frac{12}{240}\right)(191.6) - 48$$

$$= 11.58$$

$$H = \frac{12}{n(n+1)}\sum n_j\left(\bar{R}_j - \bar{R}\right)^2$$

Kruskal-Wallis *H*-Test Solution

- H_0: Identical Distrib.
- H_a: At Least 2 Differ
- α = .05
- *df* = *p* – 1 = 3 – 1 = 2
- **Critical Value(s):**



Test Statistic:
H = 11.58

Decision:
Reject at α = .05

Conclusion:
There is evidence population distrib. are different

Post hoc after **Kruskal-Wallis** Test

post-hoc Nemenyi Test

$$|\bar{R}_i - \bar{R}_j| > \sqrt{\frac{1}{C}\,\chi^2_{k-1;\alpha}\left[\frac{n(n+1)}{12}\right]\left[\frac{1}{n_i} + \frac{1}{n_j}\right]}$$

$$C = 1 - \frac{\sum_{i=1}^{i=r}(t_i^3 - t_i)}{n^3 - n},$$

with t_i the number of ties of the i-th group of ties.